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DISCREPANCY METHOD FOR SOLVING MULTISTAGE MINIMIZATION PROBLEM

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Abstract. In the paper we consider the discrepancy method for solving unstable multistage minimization problems.

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We shall consider the following problem of successive minimization, or, as one calls it, multistage minimization problem. Let the functions $f_1(x), \dots, f_p(x)$, $g_1(x), \dots, g_s(x)$ are defined on some set X_0. At the first step we solve the minimization problem on the set X_{0*} where

$$X_{0*} = \{x \in X_0 : g_j(x) \le 0,\ j = 1, \dots, m;\ g_j(x) = 0,\ j = m+1, \dots, s\} \neq \varnothing, \quad (1)$$

and find

$$f_{1*} = \inf_{X_{0*}} f_1(x); \qquad X_{1*} = \{x \in X_{0*} : f_1(x) = f_{1*}\}. \quad (2_1)$$

Suppose that $f_{1*} > -\infty$, $X_{1*} \neq \varnothing$. If at some $(k-1)$-th step $(k \ge 2)$, $f_{k-1*} > -\infty$, $X_{k-1*} \neq \varnothing$ are already known, we then find

$$f_{k*} = \inf_{X_{k-1*}} f_k(x); \qquad X_{k*} = \{x \in X_{k-1*} : f_k(x) = f_{k*}\}. \quad (2_k)$$

e.t.c. At the end, at the last step, we find

$$f_{p*} = \inf_{X_{p-1*}} f_p(x); \qquad X_{p*} = \{x \in X_{p-1*} : f_p(x) = f_{p*}\}. \quad (2_p)$$

The formulated p-stage minimization problem (1), (2), in which we look for f_{p*} and at least one point $x_{p*} \in X_{p*}$, appears in the game theory and in operations research [1–8]. As an example of the two-stage minimization problem we can take the problem of normal solution of the ordinary minimization problem.

Let us suppose that instead of the exact $f_i(x)$, $g_j(x)$, we known only their approximation $f_{i\delta}(x)$, $g_{j\delta}(x)$, such that

$$\Delta(\delta, x) = \max\Big\{ \max_{1\le i\le p} |f_{i\delta}(x) - f_i(x)| : \max_{1\le j\le s} |g_{j\delta}(x) - g_j(x)| \Big\}, \qquad x \in X_0, \quad \delta > 0 \tag{3}$$

converges to zero. We can try then to replace the initial data $f_i(x)$, $g_j(x)$ in (1), (2) with their approximations $f_{1\delta}(x)$, $g_{j\delta}(x)$ and by analogy with (2_1)–(2_p), successively determine $f_{k\delta *}$ and the sets $X_{k\delta *}$, and take them as the approximations for f_{k*} and X_{k*}, $k = 1, 2, \ldots, p$. However, this approach that looks so natural, in general case can lead to a big error for very little errors in the initial data, because, as simple examples [5] show, problem (1), (2), generally speaking, is unstable relative to perturbations of the functions $f_i(x)$, $g_j(x)$. For this reason, for solving this problem it is necessary to apply some regularization method [3, 9–12]. In what follows, we will describe and investigate one of this methods-method of discrepancy.

Let us suppose that X_0 is a subset of some metric space $\mathcal{M}$ with metric $\rho(u, v)$ and that there exists ρ-stabilizer $\Omega(x)$. Let us remind [9, 10] that the function $\Omega(x)$ is called a ρ-stabilizer if $\Omega(x) \ge 0$ for all $x \in X_0$ and the set $X_0(c) = \{x \in X_0 : \Omega(x) \le c\}$ is precompact for all $c \in \mathbb{R}$, i.e. for every sequence (x_k) from $X_0(c)$ there exists a subsequence (x_{k_m}) that ρ-converges to a certain point of $\mathcal{M}$. Let us assume that $X_{0*} \ne \varnothing$, $f_{k*} > -\infty$, $X_{k*} \ne \varnothing$ for all $k = 1, 2, \ldots, p$ and that the errors (3) of the approximations $f_{1\delta}(x)$, $g_{j\delta}(x)$ are such that

$$\Delta(\delta, x) \le \delta(1 + \Omega(x)), \qquad x \in X_0, \ \delta > 0. \tag{4}$$

Let

$$W_{0*}(\delta) = \{x \in X_0 : g^+_{j\delta}(x) \le \delta(1 + \Omega(x)), \ j = 1, \ldots, s\},$$

where $z_j^+ = \max\{0, z_j\}$ for $j = 1, \ldots, m$, and $z_j^+ = |z_j|$ for $j = m+1, \ldots, s$. Note that by (3) and (4), $X_{0*} \subseteq W_{0*}(\delta) \ne \varnothing$ for all $\delta > 0$. At each step of the discrepancy method, one must solve two minimization problems. At the first step one should approximately solve the problem.:

$$f_{1\delta}(x) + \theta_1\Omega(x) \to \inf, \qquad x \in W_{0*}(\delta), \ \theta_1 = \theta_1(\delta) > 0$$

and determine $f_{1*}(\delta) = \inf_{W_{0*}(\delta)}[f_{1\delta}(x) + \theta_1\Omega(x)]$ with the precision $\mu_1 = \mu_1(\delta)$. Then one introduces the set $V_1(\delta) = \{x \in W_{0*}(\delta) : f_{1\delta}(x) \le f_{1*}(\delta) + \mu_1\}$, solves the minimization problem $\Omega(x) \to \inf$, $x \in V_1(\delta)$, finds $\Omega_{1*}(\delta) = \inf_{V_1(\delta)} \Omega(x)$, with the precision $\varepsilon_1 = \varepsilon_1(\delta) > 0$, and, by the same token, determines the set

$$W_{1*}(\delta) = \{x \in V_1(\delta) : \Omega(x) \le \Omega_{1*}(\delta) + \varepsilon_1\}. \tag{5_1}$$

Let $W_{1*}(\delta) \ne \varnothing$. Let us suppose that for certain $k \ge 2$: $W_{k-1*}(\delta) \ne \varnothing$. Then, solving the problem $f_{k\delta}(x) + \theta_k\Omega(x) \to \inf$, $x \in W_{k-1*}(\delta)$, $\theta_k = \theta_k(\delta) > 0$, we determine $f_{k*}(\delta) = \inf_{W_{k-1*}(\delta)}\{f_{k\delta}(x) + \theta_k\Omega(x)\}$ with the precision $\mu_k = \mu_k(\delta) > 0$

and the set $V_k(\delta) = \{x \in W_{k-1*}(\delta) : f_{k\delta}(x) \le f_{k*}(\delta) + \mu_k\}$. Next, solving the problem $\Omega(x) \to \inf$, $x \in V_k(\delta)$, we find $\Omega_{k*}(\delta) = \inf_{V_k(\delta)} \Omega(x)$ with the precision $\varepsilon_k = \varepsilon_k(\delta) > 0$ and put

$$W_{k*}(\delta) = \{x \in V_k(\delta) : \Omega(x) \le \Omega_{k*}(\delta) + \varepsilon_k\}. \tag{5_k}$$

This process is completed when the set $W_{p*}(\delta)$ is determined. For $p = 1$ the method (5) turns into the known method of discrepancy for solving unstable minimization problems [**9–13**]. In this paper, on the basis of the generalization of the method from [**13**], we propose one rule for the choice of the parameters $\theta_k(\delta)$, $\mu_k(\delta)$, $\varepsilon_k(\delta)$, that guarantees conditions $f_{k*}(\delta) > -\infty$, $V_k(\delta) \ne \varnothing$, $W_{k*}(\delta) \ne \varnothing$, $k = 1, \dots, p$, prove the convergence of the method (5) and give the estimate of the convergence rate.

THEOREM 1. *Assume that the following conditions are satisfied*:

1) *The problem* (1), (2) *has a solution, i.e.* $X_{0*} \ne \varnothing$, $f_{k*} > -\infty$, $X_{k*}(\delta) \ne \varnothing$, $k = 1, \dots, p$, *and, for each fixed* k, $1 \le k \le p$, *the minimization problem*

$$\begin{aligned} &f_k(x) \to \inf, \\ &x \in X_{k*} = \{x \in X_0 : g_j(x) \le 0,\ j = 1, \dots, m;\ \ g_j(x) = 0,\ j = m+1, \dots, s; \\ &\qquad\qquad f_i(x) - f_{i*} \le 0,\ i = 1, \dots, k-1\} \ne \varnothing. \end{aligned} \tag{6}$$

satisfies the condition of the strong consistence [**14**]: *there exist constants* $\lambda_{kj} \ge 0$, $j = 1, \dots, s$, $\mu_{ki} \ge 0$, $i = 1, \dots, k-1$, *such that*

$$f_{k*} \le f_k(x) + \sum_{j=1}^{s} \lambda_{kj} g_j^+(x) + \sum_{i=1}^{k-1} \mu_{ki} \max\{f_i(x) - f_{i*}; 0\}, \qquad x \in X_0. \tag{7}$$

2) *The set* X_0 *is known exactly, but, instead of the functions* $f_i(x)$, $g_j(x)$, *their approximations satisfying condition* (4) *are known.*

3) *The parameters* $\theta_k = \theta_k(\delta)$, $\mu_k = \mu_k(\delta)$, $\varepsilon_k = \varepsilon_k(\delta)$, *are consistent with the error* $\delta > 0$, *such that*

$$\theta_k(\delta) \ge \delta(1 + 2|\lambda_k|_1), \qquad |\lambda_k|_1 = \sum_{j=1}^{s} \lambda_{kj}, \quad k = 1, \dots, p, \tag{8}$$

$$\mu_k(\delta) > \delta(2 + \Omega_* + 2|\lambda_k|_1) + \sum_{i=1}^{s} \mu_{ki}[(2\delta + \theta_i(\delta))\Omega_* + 2\delta\mu_i(\delta) + \delta\varepsilon_i(\delta)], \tag{9}$$

$$\varepsilon_k(\delta) > \Omega_* - \Omega_{k*}(\delta), \quad k = 1, \dots, p-1; \qquad \Omega_* = \inf_{X_{p*}} \Omega(x). \tag{10}$$

Then $f_{k*} > -\infty$, $V_k(\delta) \ne \varnothing$, $W_{k*}(\delta) \ne \varnothing$, $k = 1, \dots, p-1$, *and the following estimates are valid*

$$\Omega(x) \le \Omega_* + \varepsilon_i(\delta), \qquad x \in W_{i*}(\delta), \tag{11}$$

$$\max_{1 \le j \le s} g_j(x) \le 2\delta(1 + \Omega_* + \varepsilon_i(\delta)), \qquad x \in W_{i*}(\delta), \tag{12}$$

$$-2\delta|\lambda_i|_1(1+\Omega_* + \varepsilon_i(\delta)) - \sum_{j=1}^{i-1}\mu_{ij}\beta_j \le f_i(x) - f_{i*} \le \beta_i(\delta), \quad x \in W_{i*}(\delta) \tag{13}$$

where $\beta_i(\delta) = (2\delta + \theta_i(\delta))\Omega_* + 2\delta + \mu_i(\delta) + \delta\varepsilon_i(\delta)$, $i = 1,\dots,p$.

Assume that in addition to conditions 1)–3) *the following condition is satisfied:*

4) *The set* X_0 *is* ρ*-closed, the functions* $f_i(x)$, $i = 1,\dots,p$, $g_j^+(x)$, $j = 1,\dots,s$, *are lower* ρ*-semicontinuous on* X_0, $\Omega(x)$ *is* ρ*-stabilizer,*

$$\lim_{\delta\to 0}\max_{1\le k\le p}(\theta_k(\delta)+\mu_k(\delta)) = 0, \qquad \sup_{\delta>0}\max_{1\le k\le p}\varepsilon_k(\delta) < +\infty. \tag{14}$$

Then

$$\lim_{\delta\to+0}\sup_{W_{k*}(\delta)}|f_k(x) - f_{k*}| = 0, \qquad \lim_{\delta\to+0}\sup_{W_{k*}(\delta)}\rho(x, X_{k*}) = 0 \tag{15}$$

If together with 1)–4) *the following condition holds:*

5) $\Omega(x)$ *is lower* ρ*-semicontinuous on* X_0 *and*

$$\lim_{\delta\to 0}\varepsilon_p(\delta) = 0, \tag{16}$$

then

$$\lim_{\delta\to+0}\sup_{W_{k*(\delta)}}|\Omega(x) - \Omega_*| = 0, \qquad \lim_{\delta\to+0}\sup_{W_{k*}(\delta)}\rho(x, X_{p**}) = tag17$$

where $X_{p**} = \{x \in X_{p*} : \Omega(x) = \Omega_*\}$ *is the set of the* Ω*-normal solutions of the problem* (1), (2).

Note that the condition (7) is satisfied if Lagrange's function of the problem (6), (2) has a saddle point **[14]**, and that λ_{kj} and μ_{kj} in (7) can be taken to be any estimates from above for the absolute values of Lagrange's multipliers. These estimates can be obtained using numerical methods for determining the saddle points of the problem (6) **[14, 15]**. In addition, in applied problems, Lagrange's multipliers have often physical, economical or geometrical interpretation and their estimates can be obtained from practice.

PROOF. It follows from (3) and (4) that $X_{0*} \subseteq W_{0*}(\delta) \neq \varnothing$ and

$$g_j^+(x) \le g_{j\delta}^+(x) + \delta(1+\Omega(x)) \le 2\delta(1+\Omega(x)), \qquad j = 1,\dots,s \tag{18}$$

From (7), taking into account (17), we have

$$f_{k*} \le f_k(x) + 2\delta(1+\Omega(x))\cdot|\lambda_k|_1 + \sum_{i=1}^{k-1}\mu_{ki}\max\{f_i(x) - f_{i*}; 0\},$$
$$x \in W_{0*}(\delta), \quad k = 1,\dots,p. \tag{19}$$

Note also that

$$X_{p*} \subseteq X_{p-1*} \subseteq \dots \subseteq X_{0*} \subseteq W_{0*},$$
$$f_k(x_{p*}) = f_{k*}, \quad x_{p*} \in X_{p*}, \quad k = 1,\dots,p. \tag{20}$$

From (3), (4) and the inequalities (8), (19) for $k = 1$, it follows that

$$\begin{aligned} f_{1\delta}(x) + \theta_1 \Omega(x) &\geq f_1(x) - \delta(1 + \Omega(x)) + \theta_1 \Omega(x) \\ &\geq f_1 - 2\delta|\lambda_1|_1(1 + \Omega(x)) - \delta(1 + \Omega(x)) + \theta_1 \Omega(x) \\ &\geq f_{1*} - \delta(1 + 2|\lambda_1|_1) > -\infty, \qquad \text{for all } x \in W_{0*}(\delta). \end{aligned}$$

Therefore,

$$f_{1\delta}(x) \geq f_{1*} - \delta(1 + 2|\lambda_1|_1) > -\infty. \tag{21}$$

From here, taking into account that $\Omega(x) \geq 0$, $x \in X_0$, we conclude that $V_1(\delta) \neq \varnothing$, $\Omega_{1*}(\delta) \geq 0$, $W_{1*}(\delta) \neq \varnothing$ for all $\mu_1 > 0$, $\varepsilon_1 > 0$. Let us prove that $V_1(\delta) \cap X_{p*} \neq \varnothing$ for all $\delta > 0$. More precisely, let us show that $\Omega_{1**} = V_1(\delta)$, where Ω_{1**} is the first from the following sets:

$$\Omega_{k**} = \Omega_{k**}(\nu) = \{x \in X_{p*} : \Omega(x) \leq \Omega_* + \nu\}, \qquad k = 1, \dots, p-1. \tag{22}$$

$$0 < \nu \leq \min\{A; B\} \tag{23}$$

where:

$$\begin{aligned} A &= \min_{1 \leq i \leq k} \; \delta^{-1}\Big(\mu_i - \delta(2 + \Omega_* + 2|\lambda_i|_1) - \sum_{j=1}^{i-1} \mu_{ij}\beta_j\Big) \\ B &= \min_{1 \leq i \leq k} \; (\varepsilon_i(\delta) - \Omega_* + \Omega_{i*}(\delta)) \end{aligned}$$

Note that $\Omega_{k**} \neq \varnothing$ for all $\nu > 0$ by definition of Ω_{*}. For all $y \in \Omega_{1**}$, taking into account (4) and (20)–(23), we have

$$f_{1\delta}(y) \leq f_{1*} + \delta(1 + \Omega_* + \nu) \leq f_{1*}(\delta) + \delta(1 + 2|\lambda_1|_1) + \delta(1 + \Omega_* + \nu) \leq f_{1*} + \mu_1.$$

This means that $y \in V_1(\delta)$. Hence, $\Omega_{1*} \subseteq V_1(\delta)$ for all ν from (23). Then $\Omega_{1*}(\delta) \leq \Omega(y) \leq \Omega_* + \nu$ for all $y \in \Omega_{1**}$ and setting $\nu \to +0$ we obtain $\Omega_{1*}(\delta) \leq \Omega_*$. From here and (5_1) we also obtain the estimate (11) for $i = 1$. Then, from (18) it follows (12) for $i = 1$. Furthermore, taking into account (4) and (11) for $i = 1$, we have

$$\begin{aligned} f_1(x) &\leq f_{1\delta}(x) + \delta(1 + \Omega(x)) \leq f_{1*}(\delta) + \mu_1 + \delta(1 + \Omega_* + \varepsilon_1) \\ &\leq f_{1\delta}(y) + \theta_1 \Omega(y) + \mu_1 + \delta(1 + \Omega_* + \varepsilon_1) \\ &\leq f_{1*} + \delta(1 + \Omega_* + \nu) + \theta_1(\Omega_* + \nu) + \delta(1 + \Omega_* + \varepsilon_1) + \mu_1 \\ &\qquad \text{for all } x \in W_{1*}(\delta),\ y \in \Omega_{1**}. \end{aligned}$$

From here, setting $\nu \to 0$, we obtain the right hand side inequality (13). The left hand side inequality (13) follows from (19) for $k = 1$ and from the estimate (11) for $i = 1$. Finally, for all $y \in \Omega_{1**} \subseteq V_1(\delta)$, taking into account (22), (23), we have $\Omega(y) \leq \Omega_* + \nu \leq \Omega_{1*}(\delta) + \varepsilon_1(\delta)$. This means that $\Omega_{1**} \subseteq W_{1*}(\delta)$. Suppose that for some $k \geq 2$, $V_{k-1}(\delta) \neq \varnothing$, $\Omega_{k-1*}(\delta) \leq \Omega_*$, $\Omega_{k-1**} \subseteq W_{k-1**}(\delta) \neq \varnothing$, and that estimates (11)–(13) for $i = 1, \dots, k-1$ are obtained. Reasoning as in the proof of the inequality (21) and taking into account the inductive assumption we obtain

$$f_{k\delta}(x) \geq f_{k*} - \delta(1 + 2|\lambda_k|_1) - \sum_{j=1}^{k-1} \mu_{ki}\beta_i > -\infty. \tag{24}$$

It follows from here that $V_k(\delta) \neq \varnothing$, $W_{k*}(\delta) \neq \varnothing$, for all $\mu_k > 0$, $\varepsilon_k > 0$. Furthermore, taking into account (4), (20), (22)–(24) and the inclusions $\Omega_{k**} \subseteq \Omega_{k-1**} \subseteq V_{k-1}(\delta)$, we have

$$f_{k\delta}(y) \leq f_{k*} + \delta(1 + \Omega_* + \nu) \leq f_{k*}(\delta) + \delta(1 + 2|\lambda_k|_1) + \sum_{j=1}^{k-1} \mu_{ij}\beta_j + \delta(1 + \Omega_* + \nu)$$
$$\leq f_{k*}(\delta) + \mu_k(\delta), \qquad \text{for all } y \in \Omega_{k**}.$$

This means that $\Omega_{k**} \subseteq V_k(\delta)$ for all ν from (23). Then $\Omega_{k*}(\delta) \leq \Omega(y) \leq \Omega_* + \nu$ for all $y \in \Omega_{k**}$, from where, setting $\nu \to 0$, we obtain the inequality $\Omega_{k*}(\delta) \leq \Omega_*$. From here and from (5_k), it follows the estimate (11) for $i = k$. Then, from (18), it follows the estimate (12) for $i = k$. Furthermore, with the help of (4), (5_k), (20), (22), (23), we have

$$\begin{aligned} f_k(x) &\leq f_{k\delta}(x) + \delta(1 + \Omega(x)) \leq f_{k*}(\delta) + \mu_k + \delta(1 + \Omega_* + \varepsilon_k) \\ &\leq f_{k\delta}(y) + \theta_k\Omega(y) + \mu_k + \delta(1 + \Omega_* + \varepsilon_k) \\ &\leq f_{k*} + \delta(1 + \Omega_* + \nu) + \theta_k(\Omega_* + \nu) + \mu_k + \delta(1 + \Omega_* + \varepsilon_k), \end{aligned}$$

for all $x \in W_{k*}(\delta)$, $y \in \Omega_{k**} \subseteq V_k(\delta)$. From here, putting $\nu \to 0$, we obtain the right hand side inequality (13) for $i = k$. The left hand side inequality (13) is a consequence of (19), the estimate (11) for $i = k$ and the right hand side inequalities (13). Finally, for all $y \in \Omega_{k**} \subseteq V_k(\delta)$, taking into account (22), (23), we have that $\Omega(y) \leq \Omega_* + \mu \leq \Omega_{**}(\delta) + \varepsilon_k(\delta)$. This means that $\Omega_{k**} \subseteq W_{k*}(\delta)$, for $k < p$. This completes the inductive reasonings. It follows that $f_{i*}(\delta) > -\infty$, $V_i(\delta) \neq \varnothing$, $W_{i*}(\delta) \neq \varnothing$, and that the estimates (11)–(13) hold for $i = 1, \ldots, p$.

The first equality (15) follows from (13), (14). This means that when the conditions 1)–3) and (14) of Theorem 1 are satisfied, the method (5) converges with respect to function values. Let us prove the second equality (15) which means the convergence of the method with respect to the argument, supposing the conditions 1)–4) of the theorem. To this end, fix any number k, $1 \leq k \leq p$ and denote $\psi_k(\delta) = \sup \rho(x, X_{k*})$. Let (δ_l) be a sequence such that $\delta_l > 0$, $l = 1, 2, \ldots$, and $\delta_l \to 0$. Then, for every $l \in \mathbb{N}$, there exists $x_l \in W_{k*}(\delta_k)$ such that

$$\psi_k(\delta_l) - 1/l \leq \rho(x_l, X_{k*}), \qquad l = 1, 2, \ldots. \tag{25}$$

Note that $x_l \in X_0(c_k) = \{x \in X_0 : \Omega(x) \leq c_k = \Omega_* + \sup_{\delta>0} \varepsilon_k(\delta)\}$ for all $l \in \mathbb{N}$. The set $X(c_k)$ is ρ-compact and the set X_0 is closed. so that (choosing a subsequence if it is necessary) we can assume that the sequence (x_l) converges to $x_* \in X_0$. Taking into account that the functions $g_j(x)$, $j = 1, \ldots, s$, are lower ρ-semicontinuous, one can conclude that $x_* \in X_{0*}$. The function $f_1(x)$ is lower ρ-semicontinuous too, so that, using (13), we obtain: $f_{1*} \leq f_1(x_*) \leq \lim f_l(x_l) = f_{1*}$, i.e. $f_1(x_*) = f_{1*}$, $x_* \in X_{1*}$. In the same way we can derive that $f_2(x_*) = f_{2*}$, $x_* \in X_{2*}$, …, $f_k(x_*) = f_{k*}$, $x_* \in X_{k*}$. Hence, $\lim \rho(x_l, X_{k*}) = \rho(x_*, X_{k*}) = 0$. Therefore, from (25) we have that $\lim \psi_k(\delta_l) = 0$, which implies the second equality in (15), for $k = 1, \ldots, p$.

Finally, assume that all conditions 1)–5) of Theorem 1 are satisfied. Consider again a sequence (δ_l) such that $\delta_l \to 0$ and $\delta_l > 0$. Then there exists a sequence (x_l) so that

$$\sup_{x \in W_{p*}(\delta)} |\Omega(x) - \Omega_*| - 1/l \leq \Omega(x_l) - \Omega_*, \qquad l = 1, 2, \dots . \tag{26}$$

Since $x_l \in X_0(c_p)$, $l \in \mathbb{N}$, one can assume that (x_l) converges to $x_* \in X_0$. Using that the function $\Omega(x)$ is lower ρ-semicontinuous, in the similar way as above one can derive that $\Omega(x_*) = \Omega_*$. This means that $x_* \in X_{p**}$. The first equality in (17) follows from here. The second equality in (17) can be proved in the similar way.

REMARK 1. One can have the impression that in the application of the method (5), it is necessary, together with $f_{k*}(\delta)$, $\Omega_{k*}(\delta)$ to determine all the sets $W_{k*}(\delta)$. That is not true. In fact, it is enough to find the approximations $f_{k*}(\delta) + \mu_k(\delta)$, $k = 1, \dots, p$, $\Omega_*(\delta) + \varepsilon_k(\delta)$, $k = 1, \dots, p-1$, and, on the last p-th step solve one minimization problem:

$$\begin{aligned} &\Omega(x) \to \inf, \\ &x \in V_p(\delta) = \Big\{ x \in X_0 : g_{j\delta}^+(x) \leq \delta(1 + \Omega(x)),\ j = 1, \dots, s; \\ &\qquad\qquad\qquad f_{1\delta}(x) \leq f_{i*}(\delta) + \mu_i(\delta),\ i = 1, \dots, p; \\ &\qquad\qquad\qquad \Omega(x) \leq \min_{1 \leq r \leq p-1} (\Omega_{r*}(\delta) + \varepsilon_r(\delta)) \Big\} \end{aligned} \tag{27}$$

using any suitable method for this purpose (see for ex. [14, 15]. It is enough to solve the problem (27) approximately with the precision $\varepsilon_p = \varepsilon_p(\delta) > 0$. Namely, it is satisfactory to find a point $x = x_p(\delta)$ so that

$$x_p(\delta) \in V_p(\delta), \qquad \Omega(x_p(\delta)) \leq \Omega_*(\delta) + \varepsilon_p(\delta). \tag{28}$$

If the conditions 1)–4) of Theorem 1 are satisfied, we have that

$$\lim_{\delta \to +0} f_{p\delta}(x_p(\delta)) = f_{p*}, \qquad \lim_{\delta \to +0} \rho(x_p(\delta), X_{p*}) = 0, \tag{29}$$

and in case when all conditions 1)–5) are satisfied we also have that

$$\lim_{\delta \to +0} \Omega_{p\delta}(x_p(\delta)) = \Omega_*, \qquad \lim_{\delta \to +0} \rho(x_p(\delta), X_{p**}) = 0. \tag{30}$$

This means that $x_p(\delta)$ and $f_{p\delta}(x_p(\delta))$ can be taken as approximations of the solutions of the problem (1), (2). The equalities (29), (30) also mean that the operator $\mathcal{R}_\delta$ which to input data $(f_{i\delta}(x), g_{j\delta}(x), \delta)$ corresponds a point $x_p(\delta)$, is regularizing [9].

The following theorem gives an estimate of the rate of convergence of the method (5) with respect to the argument.

THEOREM 2. *Assume that the conditions* 1)–3) *of Theorem* 1 *are satisfied and* X_0 *is a closed convex subset of a Banach space* B*; the functions* $g_j^+(x)$, $j = 1, \dots, s$,

$f_1(x), \dots, f_{k-1}(x)$, $1 < k \le p$, are convex on X_0; the function $f_k(x)$ is strictly uniform convex on X_0 with the coefficients of the convexity $\omega_k(t)$ [14]. Then $X_{k*} = X_{k+1*} = \dots = X_{p*} = \{x_{k*}\}$ and

$$\|x - x_{k*}\| \le \omega_k^{-1}\Big(2\delta|\lambda_k|_1(1+\Omega_* + \varepsilon_k(\delta)) + \sum_{i=1}^{k}(\mu_{ki}[2\delta(1+\Omega_*) + \theta_i(\delta)\Omega_* + \mu_i(\delta) + \delta\varepsilon_i(\delta)])\Big), \qquad x \in W_{k*}(\delta), \tag{31}$$

where $\omega_k^{-1}(\xi)$ is the inverse function of the function $\xi = \omega_k(t)$, $\mu_{kk} = 1$.

If all functions $f_1(x), \dots, f_p(x)$ are only convex on X_0, the function $\Omega(x)$ is strictly uniform convex on X_0 with the coefficient of convexity $\omega(t)$ and if the problem

$$\Omega(x) \to \inf,$$
$$x \in X_{p*} = \{x \in X_0 : g_j(x) \le 0,\ j = 1, \dots, m;\ \ g_j(x) = 0,\ j = m+1, \dots, s;\ f_i(x) - f_{i*} \le 0,\ i = 1, \dots, p\},$$

satisfies the condition of the strong consistence of the type (7), then the set X_{p*} consists of the unique point x_{p*} and

$$\|x - x_{p*}\| \le \omega^{-1}(2\delta|\lambda_p|_1(1+\Omega_* + \varepsilon_p(\delta))) + \varepsilon_p(\delta) + \sum_{i=1}^{p}(\mu_i[2\delta(1+\Omega_*) + \theta_i(\delta)\Omega_* + \mu_(\delta) + \delta\varepsilon_i(\delta)]), \qquad x \in W_{p*}(\delta). \tag{32}$$

PROOF. It follows from the conditions of the theorem that the sets $X_{0*}, \dots, X_{p*}$ are convex and nonempty and that the set X_{k*} consists of the unique point $\{x_{k*}\}$. Then $X_{k*} = X_{k+1*} = \dots = X_{p*} = \{x_{k*}\}$. The function

$$G_k(x) = f_k(x) + \sum_{j=1}^{s}\lambda_{kj}g_j^+(x) + \sum_{i=1}^{k-1}\mu_{ki}\max\{f_i(x) - f_{i*}; 0\}, \qquad x \in X_0$$

is strictly uniform convex on X_0 with the same coefficient of the convexity $\omega_k(t)$ as the function $f_k(x)$. In addition, $G_k(x) \ge G_k(x_{k*}) = f(x_{k*}) = f_{k*}$ for all $x \in X_0$. This means that the function $G_k(x)$ achieves minimum at the point x_k. Then, [14], for $x \in X_0$, we have

$$\omega(\|x - x_{k*}\|) \le G_k(x) - G_k(x_{k*}) = f_k(x) - f_{k*} + \sum_{j=1}^{s}\lambda_{kj}g_j^+(x) + \sum_{i=1}^{k-1}\mu_{ki}\max\{f_i(x) - f_{i*}; 0\}.$$

The estimate (31) follows from here and from the estimates (12), (13). The remaining part of the theorem can be proved in the similar way.

From the conditions (14), (16) and the estimate (32), one can conclude that $\lim_{\delta\to 0}\sup_{W_{p*}(\delta)}\|x - x_{p*}\| = 0$.

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